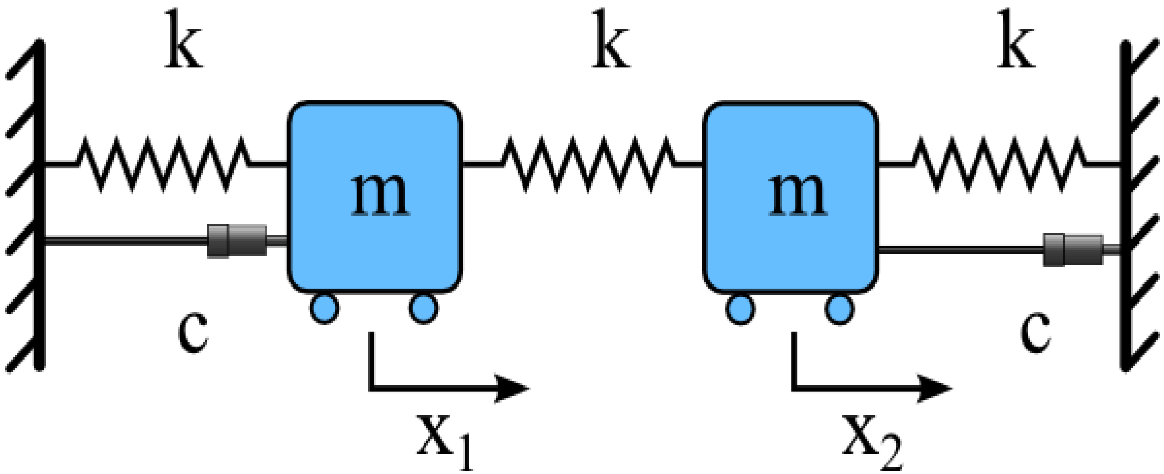
ΕΡΓΑΣΤΗΡΙΟ ΔΥΝΑΜΙΚΗΣ & ΚΑΤΑΣΚΕΥΩΝ
ΤΟΜΕΑΣ ΜΗΧΑΝΟΛΟΓΙΚΩΝ ΚΑΤΑΣΚΕΥΩΝ & ΑΥΤΟΜΑΤΟΥ ΕΛΕΓΧΟΥ
ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ
ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΑΣΚΗΣΗ 11



έκδοση ***DYNI-EXC11-2016b***

Πληροφορίες
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έστω …

διβάθμιο δυναμικό σύστημα **m-k**

$$m_1 = 4m \ \text{και} \ m_2 = m = 100kg$$

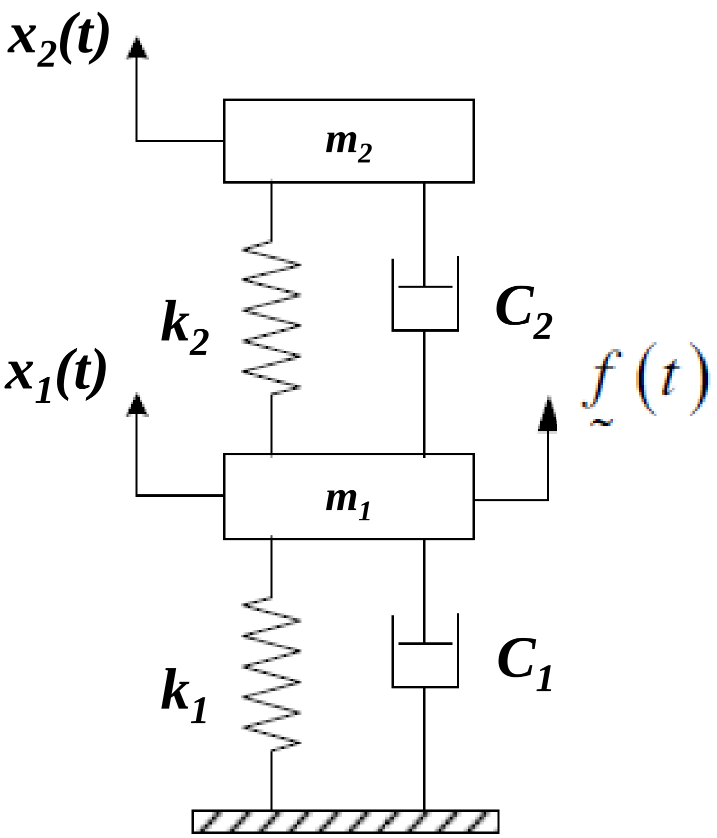
$$k_1 = k_2 = k = 400\,N/m$$

$$\underset{\sim}{f}(t) = \underset{\sim}{F}_C \cos(\Omega t) + \underset{\sim}{F}_S \sin(\Omega t)$$

όπου …

$$F_{C_1} = 1N$$

$$F_{C_2} = F_{S_1} = F_{S_2} = 0$$



? ***συνάρτηση μεταφοράς συστήματος***

για ζ=0,1 & ζ=0,4

υπολογισμός μητρώων [M], [K] & [C] του δυναμικού συστήματος με την ενεργειακή αρχή Lagrange

κινητική ενέργεια που συσσωρεύεται στις μάζες m_1 & m_2 $\rightarrow$ $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$

δυναμική ενέργεια που συσσωρεύεται στα ελατήρια k_1 & k_2 $\rightarrow$ $U = \frac{1}{2} k_1 x_1^{\,2} + \frac{1}{2} k_2 \left(x_1 - x_2 \right)^2$

η ισχύς λόγω εφαρμογής της δύναμης f_1 $\rightarrow$ $P_C = \left(\frac{1}{2} \right) c_1 \dot{x}_1^2 + \left(\frac{1}{2} \right) c_1 \left(\dot{x}_2 - \dot{x}_1 \right)^2$

η ενέργεια που διαχέεται $\rightarrow$ $P_t = f \, \dot{x}_1$

και ... $L = T - U = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \left(\frac{1}{2} k_1 x_1^{\,2} + \frac{1}{2} k_2 \left(x_1 - x_2 \right)^2 \right)$

Άσκηση 11: ΛΥΣΗ

η μαθηματική έκφραση της ενεργειακής αρχής Lagrange …

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}+\frac{\partial P_C}{\partial \dot{x}}=\frac{\partial P_t}{\partial \dot{x}} \qquad ❶$$

για την κινηματική μεταβλητή $q = x_1$ …

$$\frac{\partial L}{\partial \dot{x}_1}=\frac{\partial}{\partial \dot{x}_1}\left(\frac{1}{2}m_1\,\dot{x}_1^2+\frac{1}{2}m_2\,\dot{x}_2^2-\left(\frac{1}{2}k_1 x_1^{\,2}+\frac{1}{2}k_2\left(x_1-x_2\right)^2\right)\right)\Rightarrow\frac{\partial L}{\partial \dot{x}_1}=m_1\,\dot{x}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)=\frac{d}{dt}\left(m_1\,\dot{x}_1\right)=m_1\,\ddot{x}_1$$

$$-\left(\frac{\partial L}{\partial x_1}\right)=-\left(\frac{\partial}{\partial x_1}\left(\frac{1}{2}m_1\,\dot{x}_1^2+\frac{1}{2}m_2\,\dot{x}_2^2-\left(\frac{1}{2}k_1 x_1^{\,2}+\frac{1}{2}k_2\left(x_1-x_2\right)^2\right)\right)\right)=k_1x_1+k_2\left(x_1-x_2\right)=\left(k_1+k_2\right)x_1-k_2x_2$$

$$\frac{\partial P_C}{\partial \dot{x}_1}=\frac{\partial}{\partial \dot{x}_1}\left(\left(\frac{1}{2}\right)c_1\dot{x}_1^2+\left(\frac{1}{2}\right)c_2\left(\dot{x}_2-\dot{x}_1\right)^2\right)=c_1\dot{x}_1+c_2\left(\dot{x}_2-\dot{x}_1\right)\left(-1\right)=\left(c_1+c_2\right)\dot{x}_1-c_2\dot{x}_2$$

$$\frac{\partial P_t}{\partial \dot{x}_1}=f$$

οπότε… ❶ $\rightarrow$ $$m_1\ddot{x}_1+(k_1+k_2)x_1-k_2x_2+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2=f\Rightarrow$$

$$\Rightarrow m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=f$$ ❷

για την κινηματική μεταβλητή q = x_2 …

$$\frac{\partial L}{\partial \dot{x}_2}=\frac{\partial}{\partial \dot{x}_2}\left(\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2-\left(\frac{1}{2}k_1x_1^{\,2}+\frac{1}{2}k_2(x_1-x_2)^2\right)\right)=m_2\dot{x}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)=\frac{d}{dt}(m_2\dot{x}_2)=m_2\ddot{x}_2$$

$$-\left(\frac{\partial L}{\partial x_2}\right)=-\left(\frac{\partial}{\partial x_2}\left(\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2-\left(\frac{1}{2}k_1x_1^{\,2}+\frac{1}{2}k_2(x_1-x_2)^2\right)\right)\right)=k_2(x_1-x_2)(-1)=-k_2x_1+k_2x_2$$

$$\frac{\partial P_C}{\partial \dot{x}_2}=\frac{\partial}{\partial \dot{x}_2}\left(\left(\frac{1}{2}\right)c_1\dot{x}_1^2+\left(\frac{1}{2}\right)c_2(\dot{x}_2-\dot{x}_1)^2\right)=c_2(\dot{x}_2-\dot{x}_1)$$ ***και…*** $$\frac{\partial P_t}{\partial \dot{x}_2}=0$$

οπότε… ❶ $\rightarrow$ $m_2\ddot{x}_2 - k_2x_1 + k_2x_2 + c_2\left(\dot{x}_2 - \dot{x}_1\right) = 0 \Rightarrow$

$$m_2\ddot{x}_2 + c_2\left(\dot{x}_2 - \dot{x}_1\right) - k_2x_1 + k_2x_2 = 0 \quad ❸$$

μητρωϊκή μορφή των ❷ ***και*** ❸ ***…***

$$\left\{\begin{matrix} m_1\ddot{x}_1 + \left(c_1 + c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2x_2 = f \\ m_2\ddot{x}_2 + c_2\left(\dot{x}_2 - \dot{x}_1\right) - k_2x_1 + k_2x_2 = 0 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{matrix} \begin{bmatrix} m_1 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \left(k_1 + k_2\right) & -k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = f \\ \begin{bmatrix} 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \end{matrix}\right\} \Rightarrow$$

$$\ldots \Rightarrow \underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\underline{M}} \underbrace{\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}}_{\underline{\ddot{x}}} + \underbrace{\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}}_{\underline{C}} \underbrace{\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}}_{\underline{\dot{x}}} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{\underline{K}} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}}_{\underline{x}} = \underbrace{\begin{Bmatrix} f \\ 0 \end{Bmatrix}}_{\underline{F}}$$

αντικαθιστώντας τα δεδομένα ...

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}$$

οπότε τα μητρώα ...

$$\underline{M} = \begin{bmatrix} 4m & 0 \\ 0 & m \end{bmatrix} \qquad \underline{K} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \qquad \underline{C} = \begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix}$$

Άσκηση 11: ΛΥΣΗ

εξίσωση ισορροπίας ...

$$\underline{M}\,\ddot{\underset{\sim}{x}} + \underline{C}\,\dot{\underset{\sim}{x}} + \underline{K}\,\underset{\sim}{x} + = \underset{\sim}{f}$$

όπου ...

η διεγείρουσα δύναμη είναι: $\underset{\sim}{f}(t) = \underset{\sim}{F}_C \cos(\Omega t) + \underset{\sim}{F}_S \sin(\Omega t)$

η μόνιμη απόκριση είναι: $\underset{\sim}{x}(t) = \underset{\sim}{x}_P(t) = \underset{\sim}{X}_C \cos(\Omega t) + \underset{\sim}{X}_S \sin(\Omega t)$

1η χρονική παράγωγος: $\dot{\underset{\sim}{x}}(t) = \dot{\underset{\sim}{x}}_P(t) = -\Omega \underset{\sim}{X}_C \sin(\Omega t) + \Omega \underset{\sim}{X}_S \cos(\Omega t)$

2η χρονική παράγωγος: $\ddot{\underset{\sim}{x}}(t) = \ddot{\underset{\sim}{x}}_P(t) = -\Omega^2 \underset{\sim}{X}_C \cos(\Omega t) - \Omega^2 \underset{\sim}{X}_S \sin(\Omega t)$

$\Rightarrow$

Άσκηση 11: ΛΥΣΗ

$$\ldots \Rightarrow \underline{M}\left(-\Omega^2 \underset{\sim}{X}_C \cos(\Omega t) - \Omega^2 \underset{\sim}{X}_S \sin(\Omega t)\right) + \underline{C}\left(-\Omega \underset{\sim}{X}_C \sin(\Omega t) + \Omega \underset{\sim}{X}_S \cos(\Omega t)\right)$$

$$+\underline{K}\left(\underset{\sim}{X}_C \cos(\Omega t) + \underset{\sim}{X}_S \sin(\Omega t)\right) = \underset{\sim}{F}_C \cos(\Omega t) + \underset{\sim}{F}_S \sin(\Omega t)$$

$$\vdots$$

$$\Downarrow$$ ***ομαδοποίηση συν- & ημι- τονικών όρων***

$$\left(-\Omega^2 \underline{M}\underset{\sim}{X}_C + \Omega \underline{C}\underset{\sim}{X}_S + \underline{K}\underset{\sim}{X}_C - \underset{\sim}{F}_C\right)\cos(\Omega t) + \left(-\Omega^2 \underline{M}\underset{\sim}{X}_S - \Omega \underline{C}\underset{\sim}{X}_C + \underline{K}\underset{\sim}{X}_S - \underset{\sim}{F}_S\right)\sin(\Omega t) = 0$$

$$\Downarrow$$

ΠΡΕΠΕΙ να ισχύει για κάθε t

$$\Downarrow$$

$$\begin{Bmatrix} \left(-\Omega^2 \underline{M}\underset{\sim}{X}_C + \Omega \underline{C}\underset{\sim}{X}_S + \underline{K}\underset{\sim}{X}_C - \underset{\sim}{F}_C\right) = \underset{\sim}{0} \\ \left(-\Omega^2 \underline{M}\underset{\sim}{X}_S - \Omega \underline{C}\underset{\sim}{X}_C + \underline{K}\underset{\sim}{X}_S - \underset{\sim}{F}_S\right) = \underset{\sim}{0} \end{Bmatrix} \Rightarrow \begin{Bmatrix} -\Omega^2 \underline{M}\underset{\sim}{X}_C + \Omega \underline{C}\underset{\sim}{X}_S + \underline{K}\underset{\sim}{X}_C = \underset{\sim}{F}_C \\ -\Omega^2 \underline{M}\underset{\sim}{X}_S - \Omega \underline{C}\underset{\sim}{X}_C + \underline{K}\underset{\sim}{X}_S = \underset{\sim}{F}_S \end{Bmatrix} \Rightarrow$$

$$\Rightarrow \begin{Bmatrix} \left(-\Omega^2 \underline{M} + \underline{K}\right)\underset{\sim}{X}_C + \Omega \underline{C}\underset{\sim}{X}_S = \underset{\sim}{F}_C \\ -\Omega \underline{C}\underset{\sim}{X}_C + \left(-\Omega^2 \underline{M} + \underline{K}\right)\underset{\sim}{X}_S = \underset{\sim}{F}_S \end{Bmatrix} \Rightarrow \begin{bmatrix} \left(-\Omega^2 \underline{M} + \underline{K}\right) & \Omega \underline{C} \\ -\Omega \underline{C} & \left(-\Omega^2 \underline{M} + \underline{K}\right) \end{bmatrix} \begin{bmatrix} \underset{\sim}{X}_C \\ \underset{\sim}{X}_S \end{bmatrix} = \begin{bmatrix} \underset{\sim}{F}_C \\ \underset{\sim}{F}_S \end{bmatrix}$$

$$\ldots \Rightarrow \begin{bmatrix} -\Omega^2 \begin{bmatrix} 4m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} & \Omega \begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix} \\ -\Omega \begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix} & -\Omega^2 \begin{bmatrix} 4m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \end{bmatrix} \begin{bmatrix} X_{C_1} \\ X_{C_2} \\ X_{S_1} \\ X_{S_2} \end{bmatrix} = \begin{bmatrix} F_{C_1} \\ F_{C_2} \\ F_{S_1} \\ F_{S_2} \end{bmatrix}$$

$$\ldots \Rightarrow$$

$$\begin{bmatrix} \begin{bmatrix} -\Omega^2(4m+2k) & \Omega^2 k \\ \Omega^2 k & -\Omega^2(m+k) \end{bmatrix} & \begin{bmatrix} 2\Omega c & -\Omega c \\ -\Omega c & \Omega c \end{bmatrix} \\ \begin{bmatrix} -2\Omega c & \Omega c \\ \Omega c & -\Omega c \end{bmatrix} & \begin{bmatrix} -\Omega^2(4m+2k) & \Omega^2 k \\ \Omega^2 k & -\Omega^2(m+k) \end{bmatrix} \end{bmatrix} \begin{bmatrix} X_{C_1} \\ X_{C_2} \\ X_{S_1} \\ X_{S_2} \end{bmatrix} = \begin{bmatrix} F_{C_1} \\ F_{C_2} \\ F_{S_1} \\ F_{S_2} \end{bmatrix} \Rightarrow$$

… $\Rightarrow$

$$\begin{bmatrix} -\Omega^2(4m+2k) & \Omega^2 k & 2\Omega c & -\Omega c \\ \Omega^2 k & -\Omega^2(m+k) & -\Omega c & \Omega c \\ -2\Omega c & \Omega c & -\Omega^2(4m+2k) & \Omega^2 k \\ \Omega c & -\Omega c & \Omega^2 k & -\Omega^2(m+k) \end{bmatrix} \begin{bmatrix} X_{C_1} \\ X_{C_2} \\ X_{S_1} \\ X_{S_2} \end{bmatrix} = \begin{bmatrix} F_{C_1} \\ F_{C_2} \\ F_{S_1} \\ F_{S_2} \end{bmatrix}$$

… $\Rightarrow$

$$\begin{bmatrix} X_{C_1} \\ X_{C_2} \\ X_{S_1} \\ X_{S_2} \end{bmatrix} = \begin{bmatrix} -\Omega^2(4m+2k) & \Omega^2 k & 2\Omega c & -\Omega c \\ \Omega^2 k & -\Omega^2(m+k) & -\Omega c & \Omega c \\ -2\Omega c & \Omega c & -\Omega^2(4m+2k) & \Omega^2 k \\ \Omega c & -\Omega c & \Omega^2 k & -\Omega^2(m+k) \end{bmatrix}^{-1} \begin{bmatrix} F_{C_1} \\ F_{C_2} \\ F_{S_1} \\ F_{S_2} \end{bmatrix}$$

όπου … $c = 2\zeta\Omega m$

$\Rightarrow$

$$\ldots \Rightarrow$$

$$\begin{bmatrix} X_{C_1} \\ X_{C_2} \\ X_{S_1} \\ X_{S_2} \end{bmatrix} = \begin{bmatrix} -1200\Omega^2 & 400\Omega^2 & 400\zeta\Omega^2 & -200\zeta\Omega^2 \\ 400\Omega^2 & -500\Omega^2 & -200\zeta\Omega^2 & 200\zeta\Omega^2 \\ 400\zeta\Omega^2 & 200\zeta\Omega^2 & -1200\Omega^2 & 400\zeta\Omega^2 \\ 200\zeta\Omega^2 & -200\zeta\Omega^2 & 400\zeta\Omega^2 & -500\Omega^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

... επομένως

$$X_{C_1} = \frac{\begin{vmatrix} 1 & 400\Omega^2 & 400\zeta\Omega^2 & -200\zeta\Omega^2 \\ 0 & -500\Omega^2 & -200\zeta\Omega^2 & 200\zeta\Omega^2 \\ 0 & 200\zeta\Omega^2 & -1200\Omega^2 & 400\zeta\Omega^2 \\ 0 & -200\zeta\Omega^2 & 400\zeta\Omega^2 & -500\Omega^2 \end{vmatrix}}{\det \begin{vmatrix} -1200\Omega^2 & 400\Omega^2 & 400\zeta\Omega^2 & -200\zeta\Omega^2 \\ 400\Omega^2 & -500\Omega^2 & -200\zeta\Omega^2 & 200\zeta\Omega^2 \\ 400\zeta\Omega^2 & 200\zeta\Omega^2 & -1200\Omega^2 & 400\zeta\Omega^2 \\ 200\zeta\Omega^2 & -200\zeta\Omega^2 & 400\zeta\Omega^2 & -500\Omega^2 \end{vmatrix}}$$

... και

$$X_{C_2} = \frac{\begin{vmatrix} -1200\Omega^2 & 1 & 400\zeta\Omega^2 & -200\zeta\Omega^2 \\ 400\Omega^2 & 0 & -200\zeta\Omega^2 & 200\zeta\Omega^2 \\ 400\zeta\Omega^2 & 0 & -1200\Omega^2 & 400\zeta\Omega^2 \\ 200\zeta\Omega^2 & 0 & 400\zeta\Omega^2 & -500\Omega^2 \end{vmatrix}}{\det \begin{vmatrix} -1200\Omega^2 & 400\Omega^2 & 400\zeta\Omega^2 & -200\zeta\Omega^2 \\ 400\Omega^2 & -500\Omega^2 & -200\zeta\Omega^2 & 200\zeta\Omega^2 \\ 400\zeta\Omega^2 & 200\zeta\Omega^2 & -1200\Omega^2 & 400\zeta\Omega^2 \\ 200\zeta\Omega^2 & -200\zeta\Omega^2 & 400\zeta\Omega^2 & -500\Omega^2 \end{vmatrix}}$$

... ομοίως X_{S_1} και X_{S_2}

... στη συνέχεια από $F_j = \sqrt{F_{C_j}^2 + F_{S_j}^2}$ ... j =1, 2

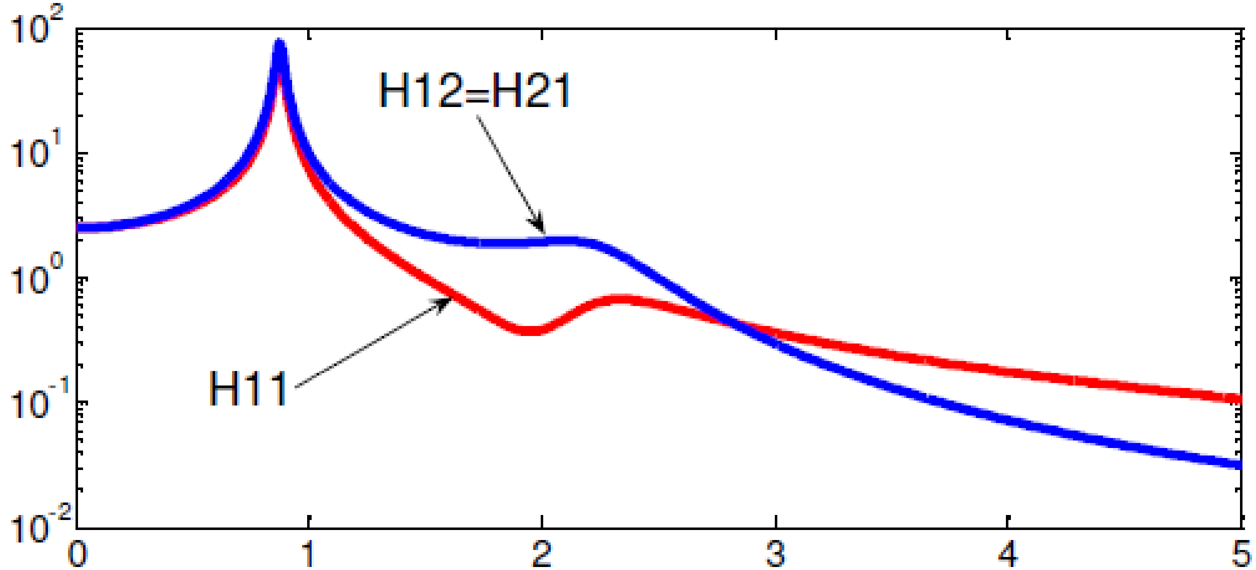
... και $X_i = \sqrt{X_{C_i}^2 + X_{S_i}^2}$ ... i =1, 2

$$\Rightarrow \quad H_{ij} = \left(\frac{X_i}{F_j} \right)$$

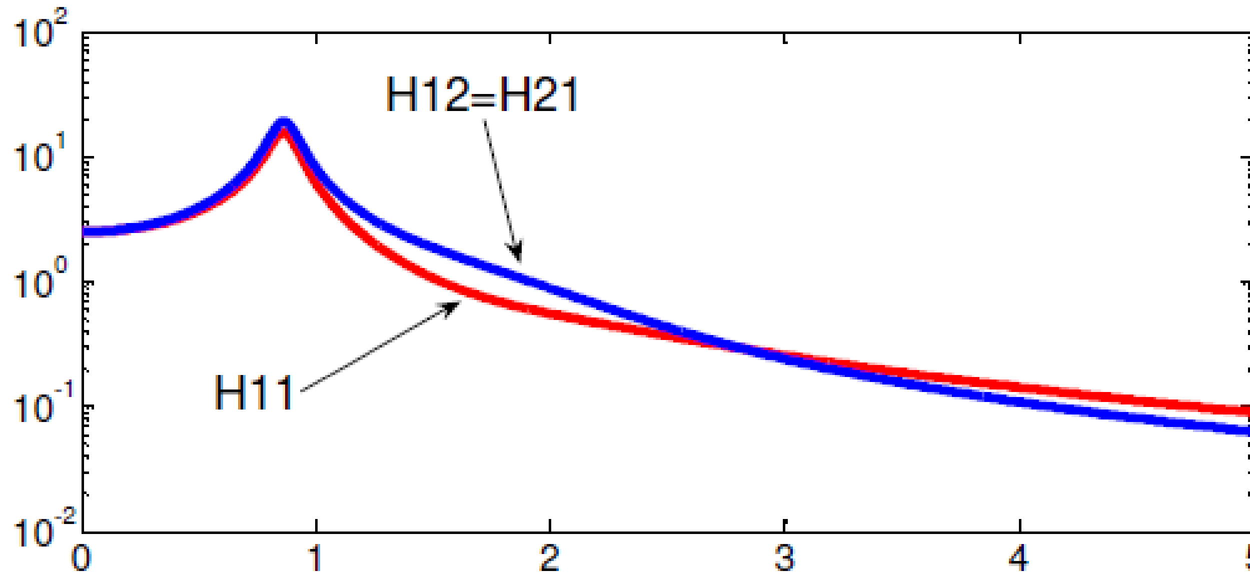
η συνάρτηση μεταφοράς H_{22} δεν ορίζεται επειδή η δύναμη διέγερσης $F_2 = 0$

συναρτήσεις μεταφοράς για ...

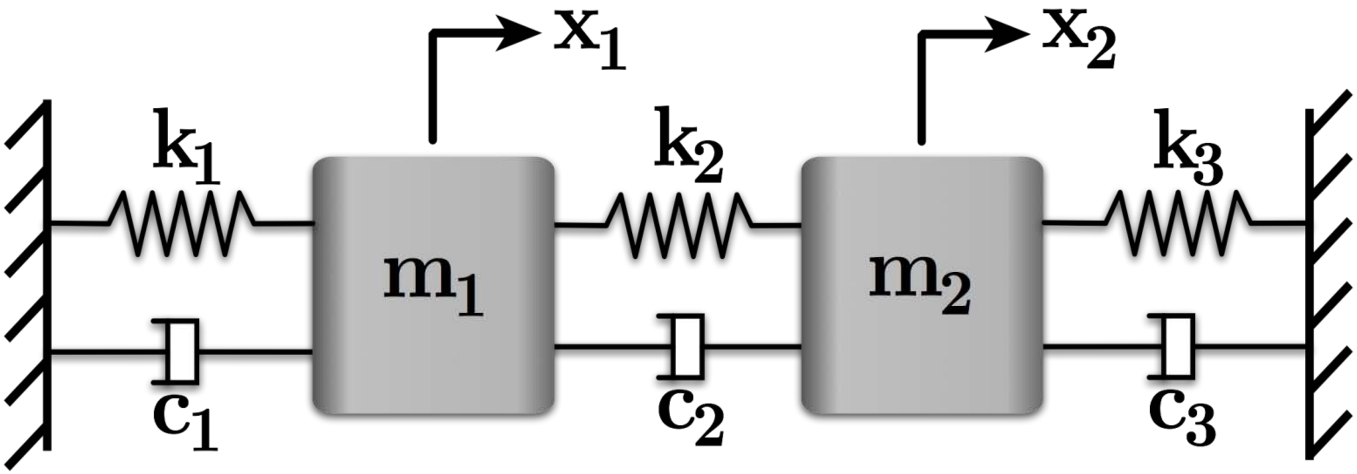
ζ=0,1



ζ=0,4



Ευχαριστώ για την προσοχή σας!



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